THEORETICAL CONDITIONS FOR MODELLING THE THERMALLY STRESSED STATE OF SOLIDS

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We obtain and analyze a complete set of similarity criteria which serve to insure similarity in a solid of arbitrary shape, actual or model, simple or composite, when the thermally stressed state of the solid is modelled under various conditions of thermal loading.

We study the problem of modelling the thermally stressed state of a solid, initially at a uniform temperature T_0 and zero stresses and deformations; the thermal stresses arise as the result of a thermal interaction with an external medium whose temperature T_1 varies with the time for the case of free and forced convection. The temperature field which arises in the body is accompanied therein by nonstationary thermal stresses, deformations, and displacements.

The system of fundamental equations is as follows.

1. The heat conduction equation and associated boundary conditions:

$$\frac{\partial T}{\partial t} = a\nabla^2 T, \tag{1}$$

$$\mathrm{grad}_n T = -\frac{\alpha}{\lambda}(T_1 - T). \tag{2}$$

2. The equation of motion

$$\frac{\partial \sigma_{ij}}{\partial x_i} - \rho\frac{\partial^2 u_i}{\partial t^2} = 0. \tag{3}$$

3. The equations relating the relative deformation components (ε_{ij}) and the displacements (u_i, u_j)

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{4}$$

4. Equations relating the stresses and deformations for the following cases:

a) a linearly elastic Hooke's Law material

$$\varepsilon_{ij} = \beta T\delta_{ij} + \frac{1}{2G}\left(\sigma_{ij} - \frac{\mu}{1+\mu}\sigma_{kk}\delta_{ij}\right); \tag{5}$$

b) a material exhibiting simple viscoelastic properties: a Voight or Maxwell body [2]

$$\sigma_{ij} = 2G\varepsilon_{ij} + G\tau_1\dot{\varepsilon}_{ij} + \delta_{ij}\left[\left(K - \frac{2}{3}G\right)\varepsilon_{kk} - \frac{2}{3}G\,\tau_1\dot{\varepsilon}_{kk} - 3K\beta T\right]; \tag{6}$$

$$\dot{\sigma}_{ij} + \tau_1^{-1}\sigma_{ij} = 2G\dot{\varepsilon}_{ij} + \delta_{ij}\left[K\varepsilon_{kk}\tau_1^{-1} + \left(K - \frac{2}{3}G\right)\dot{\varepsilon}_{kk} + 3K\beta(\dot{T} + \tau_1^{-1}T)\right]; \tag{7}$$

c) materials with more involved viscoelastic properties, represented by parallel and series combinations of Voight and Maxwell bodies, the elastic and viscous elements of which are characterized, respectively, by stiffness constants $G_1, G_2, \ldots, G_r$ and viscosities $\eta_1, \eta_2, \ldots, \eta_n$ [3].

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TABLE 1. Similarity Criteria for the Thermally Stressed State of a Solid Both Full-Scale and Model

Group I	Material properties for part A of the model
Linear elastic material	$\mu_A,\ \left[\frac{(E_0)_A\, l_A^2}{\rho_A\, a_A^2}\right],\ \frac{(A_1)_A}{(E_0)_A\,\beta_A},\ \frac{(A_2)_A}{(E_0)_A\,\beta^2},\ \ldots,\ \frac{(A_m)_A}{(E_0)_A\,\beta_A^m}$
Material with simple linear viscoelastic properties (Voight or Maxwell body)	$\frac{(G_{01})_A}{K_A},\ \frac{(\eta_{01})_A}{(G_{01})_A\,\rho_A\, l_A^2},\ \frac{(\eta_{01})_A}{a_A\,\rho_A},\ \frac{(A_1)_A}{(G_{01})_A\,\beta_A},\ \ldots,\ \frac{(A_m)_A}{(G_{01})_A\,\beta_A^m}$
Material with complex linear viscoelastic properties	$\frac{(G_{01})_A}{K_A},\ \frac{(\eta_{01})_A}{(G_{01})_A\,\rho_A\, l_A^2},\ \frac{(\eta_{01})_A}{a_A\,\rho_A},\ \frac{(A_1)_A}{(G_{01})_A\,\beta_A},\ \ldots,\ \frac{(A_m)_A}{(G_{01})_A\,\beta_A^m},$ $\frac{(\eta_{02})_A}{(\eta_{01})_A},\ \ldots,\ \frac{(\eta_{0n})_A}{(\eta_{01})_A},\ \frac{(G_{02})_A}{(G_{01})_A},\ \ldots,\ \frac{(G_{0r})_A}{(G_{01})_A}$
Group II	Material properties for part B of the model
Material A elastic or viscoelastic; material B elastic	$\mu_B,\ \frac{\beta_B}{\beta_A},\ \frac{\lambda_B}{\lambda_A},\ \frac{a_B}{a_A},\ \left[\frac{\rho_B}{\rho_A}\right],\ \frac{(G_{01})_B}{(G_{01})_A},$ $\frac{l_B}{l_A},\ \frac{(A_1)_B}{(A_1)_A},\ \ldots,\ \frac{(A_k)_B}{(A_k)_A}$
Material A and B viscoelastic	$\frac{(G_{01})_B}{K_B},\ \frac{\beta_B}{\beta_A},\ \frac{\lambda_B}{\lambda_A},\ \frac{a_B}{a_A},\ \left[\frac{\rho_B}{\rho_A}\right],$ $\frac{(G_{01})_B}{(G_{01})_A},\ \frac{(A_1)_B}{(A_1)_A},\ \ldots,\ \frac{(A_k)_B}{(A_k)_A},\ \frac{l_B}{l_A},$ $\frac{(\eta_{01})_B}{(\eta_{01})_A},\ \ldots,\ \frac{(\eta_{0i})_B}{(\eta_{0i})_A},\ \ldots,\ \frac{(G_{01})_B}{(G_{01})_A},\ \ldots,\ \frac{(G_{0j})_B}{(G_{0j})_A}$
Group III	Conditions defining the thermal interaction with the external medium
$T_1 = \mathrm{const}$ (thermostatics)	$\beta\Delta T_0,\ \frac{\Delta T_0}{T_0},\ \mathrm{Bi}$ { GrPr, $\frac{\lambda_c}{\lambda_A},\ \frac{a_c}{a_A},\ \frac{l_c}{l_A}$ — for free convection; Re, Pr, $\frac{\lambda_c}{\lambda_A},\ \frac{a_c}{a_A},\ \frac{l_c}{l_A}$ — for forced convection }
Linear law $T_1 = f(t)$	$\beta_A\Delta T_0,\ \frac{\Delta T_0}{T_0},\ \mathrm{Bi}\{\ \},\ \frac{bl_A^2}{a_A\,\Delta T_0}$
Harmonic temperature oscillations of the medium	$\beta_A\Delta T_0,\ \frac{\Delta T_0}{T_0},\ \mathrm{Fo},\ \frac{\Delta T_{1\max}}{\Delta T_0},\ \mathrm{Bi}\{\ \}$
Temperature of medium varying according to an arbitrary law	$\beta_A\Delta T_0,\ \frac{\Delta T_0}{T_0};\ \mathrm{Bi}\{\ \},\ \frac{b_1 l_A^2}{a\Delta T_0},\ \ldots,\ \frac{b_n l_A^{2n}}{a_A^n\,\Delta T_0}$

Results which follow satisfaction of the criteria (formulas for transition from model to full-scale)

$$\sigma_{ij} = \sigma'_{ij}\,\frac{G_{01}}{G'_{01}},\quad u_i = u'_i\,\frac{l}{l'},\quad \varepsilon_{ij} = \varepsilon'_{ij}$$

$$\text{for } x_i = x'_i\,\frac{l}{l'} \text{ and } t = t'\,\frac{a\,(l')^2}{a'\,l^2}$$

The initial conditions are:

$$\text{for } t=0,\ T=T_0,\ \sigma_{ij}=0,\ \varepsilon_{ij}=0,\ u_i=0. \qquad (8)$$

We describe the temperature dependence of the modulus of elasticity in the general case by a polynomial:

$$E(T) = E_0 + A_1T + A_2T^2 + \ldots + A_mT^m. \qquad (9)$$

The dependence of an arbitrary physical constant on the temperature can also be represented in the form (9) (values of constants with zero subscripts will be those at temperature T_0). If the temperature of the surrounding medium varies with the time, the relations (1)-(9) are supplemented with the corresponding time function $T_1 = f(t)$.

In the case of a composite body consisting of parts A and B, additional relations are needed for body B, including the relations (1)-(9). Thermal and stress boundary conditions acting at the contact boundary S between the bodies A and B are also required:

$$\lambda_A (\mathrm{grad}_n T)_A = -\lambda_B (\mathrm{grad}_n T)_B, \quad (10)$$

$$(\sigma_n)_A = (\sigma_n)_B, \quad (11)$$

$$(u_n)_A = (u_n)_B, \quad (12)$$

$$(u_\tau)_A = (u_\tau)_B. \quad (13)$$

By subjecting these equations and the associated boundary conditions to similarity theory methods we obtain a complete set of similarity criteria for the various cases of thermal loading and assumptions relating to the mechanical properties of the material bodies. Moreover the concept of similarity of thermally stressed states is particularized in terms of the similarity of stress, deformation, and displacement fields.

For example, in the case of a homogeneous body subjected to harmonic temperature oscillations (of amplitude $\Delta T_{1\max}$ and of period t_0) in the external medium undergoing forced motion, the complete set of similarity criteria for a solid with simple viscoelastic properties is as follows:

$$\frac{G_{01}}{K}, \ \frac{\eta_{01}^2}{G_{01}\rho l^2}, \ \frac{\eta_{01}}{a\rho}, \ \frac{A_1}{G_{01}\beta}, \ \frac{A_2}{G_{01}\beta^2}, \ \ldots, \ \frac{A_m}{G_{01}\beta^m},$$
$$\beta\Delta T_0, \ \frac{\Delta T_{1\max}}{\Delta T_0}, \ \mathrm{Fo}, \quad (14)$$
$$\mathrm{Bi}\left\{\mathrm{Re}, \ \mathrm{Pr}, \ \frac{\lambda_c}{\lambda}, \ \frac{a_c}{a}, \ \frac{l_c}{l}\right\}.$$

An analysis of the structure of the complete systems of similarity criteria shows that each such system can be subdivided into two or three autonomous groups. In the first group we place criteria relating the mechanical and thermophysical characteristics of the material, both full-scale and model. In the criteria of the second group we have the ratios of the mechanical and thermophysical characteristics of the bodies A and B, these characteristics being essential for the process being investigated, and also the ratios of their linear dimensional characteristics. Criteria in the third group define the conditions for the thermal interaction of the solid with the external medium in model experiments.

All the criteria for characterizing the similarity of the thermally stressed states in the full-scale body and its model are presented in Table 1; this makes it possible to form a complete set of criteria for modelling temperature effects in both simple and composite solids in an arbitrary case.

The criterion $E_0 l^2/\rho a^2$ is used for comparing the speeds of propagation of sound and heat in the material of the solid. Since the first of these speeds exceeds the second by many orders of magnitude, stresses in solids appear almost instantaneously following the propagation of heat and the buildup of the temperature field. Hence the criterion $E_0 l^2/\rho a^2$ (and also the criterion ρ_B/ρ_A derived from it) is inessential and in modelling can be neglected. As a result of this simplification it becomes clear that the restrictions contained in the first group of criteria do not exclude application to a model of the full-scale material, for all the types of thermal loading considered, in the case of an elastic material. In this case the criteria relating the temperature dependence of the modulus of elasticity in the full-scale body and its model are automatically satisfied. This result can be extended to apply to modelling in which the temperature dependence of an arbitrary characteristic is taken into account.

In modelling which takes viscoelastic properties into account the choice of material for the model must obey the following additional relationships:

1) between the elastic and viscous characteristics

$$\frac{\eta_{01}^2}{G_{01}\rho l^2} = \frac{(\eta_{01}')^2}{G_{01}'\rho' (l')^2}; \quad (15)$$

2) between the viscoelastic and thermophysical characteristics

$$\frac{\eta_{01}}{a\rho} = \frac{\eta'_{01}}{a'\rho'}\,. \tag{16}$$

In addition, the parametric criteria appearing in Group I of Table 1 must be observed.

The criteria of Group II are characteristic only for composite bodies. They serve as supplementary restrictions on the choice of material in a model of the composite body.

The criteria of Group II determine the law of temperature variation $T_1 = f(t)$ in the model and also the conditions of heat transfer between the model and the external medium.

NOTATION

x_i, x_j	are Cartesian coordinates;
σ_{ij}	are stress components;
t	is the time;
ρ	is the density;
$\alpha, \beta, a, \lambda$	are coefficients of heat transfer, of linear expansion, temperature diffusivity and thermal diffusivity, respectively;
E, G, K	are moduli of elasticity, shear and compression, respectively;
μ	is the Poisson number;
η_1	is the shear viscosity coefficient;
τ_1	is the stress relaxation time;
T	is the temperature, °K;
Bi, Gr, Pr, Re	are the Biot, Grashof, Prandtl and Reynolds numbers, respectively;
ν_c	is the kinematic viscosity of medium;
g	is the free fall acceleration;
l	is the characteristic linear dimension;
w_c	is the forced velocity of medium;

Subscripts

A	refers to body A;
B	refers to body B;
c	refers to external medium.

LITERATURE CITED

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